A Minimalist's Implementation of the 3-d Divide-and-Conquer Convex Hull Algorithm

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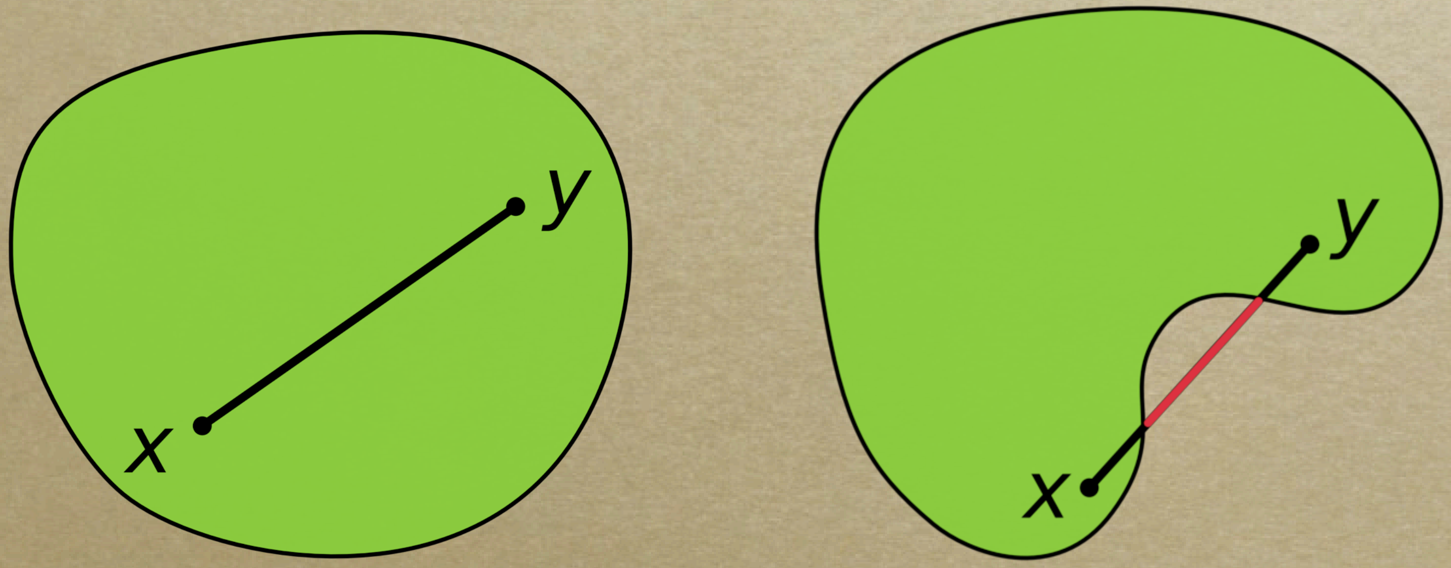
Simple Polygons

- Polygon = A consecutive set of vertices and edges that form a closed path
- Simple = No edges cross
- Have a *Boundary* and *Interior*

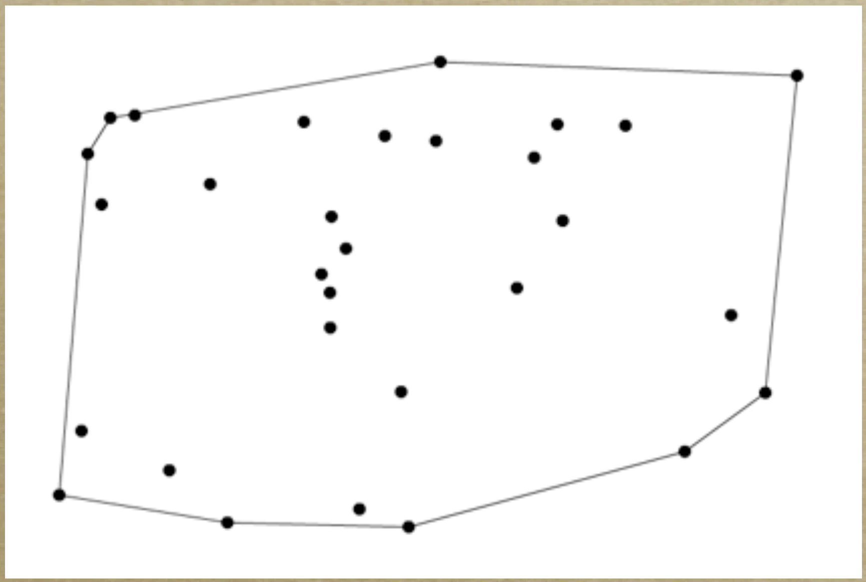
Convex Polygons

- For all pairs of points *x* and *y* inside a polygon, the line segment (*x*, *y*) does not leave the polygon



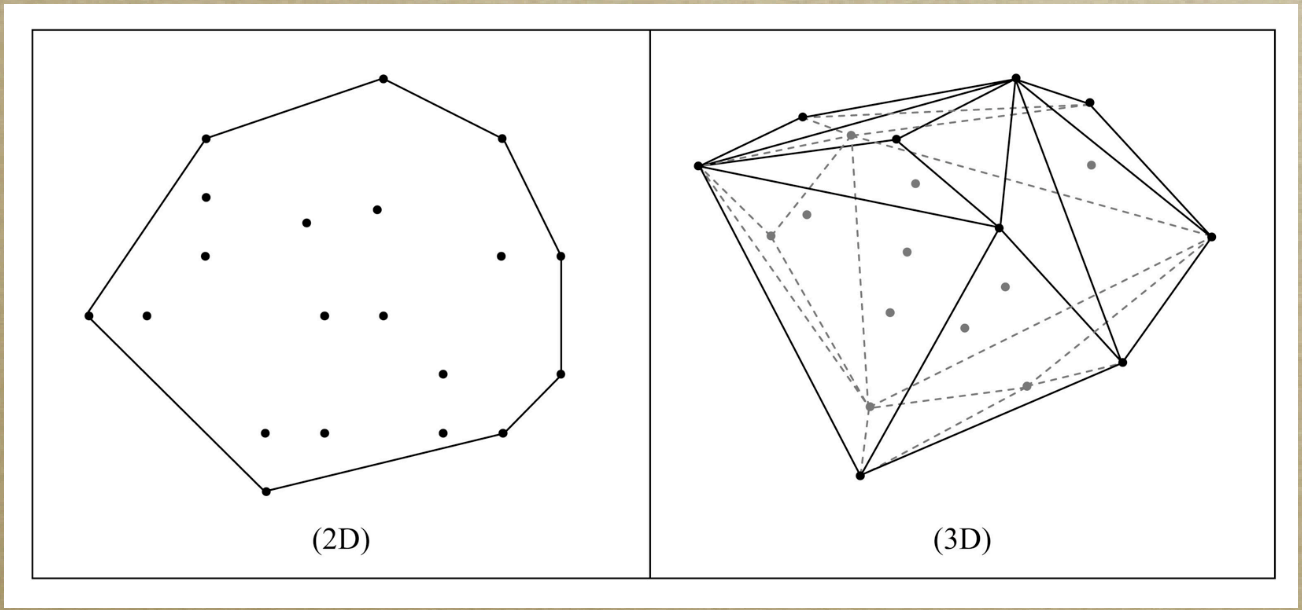
2-d Convex Hull

- Given a set *P* of points in the plane
- The smallest convex polygon that contains all of *P*



3-d Convex Hull

- Given a set *P* of points in R^3
- The smallest convex **polyhedron** that contains all of *P*

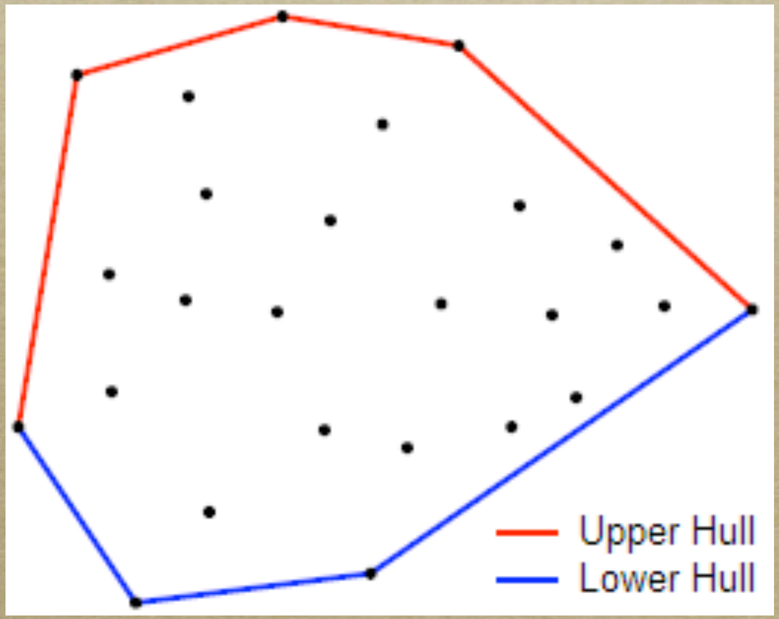


Lower/Upper Hull

- A convex hull can be decomposed into a *lower* and *upper* hull.
- Edges (2d) or Faces (3d) that can be seen from above or below
- Vertical ray (from $+\infty$)
  - enters polygon through *upper* hull
  - leaves through *lower* hull

Lower/Upper Hull

- An edge will either be in the upper hull *or* in the lower hull
- Construct a CH by finding a lower and upper hull and putting them together



Demo

Our Goal

- Given a set of points in R^3
- Construct a 3D lower convex hull
- Strategy:

Divide and Conquer

Divide and Conquer

- Given an input set S, of size n, solve some global problem on S.
- Need two things:
  - Know how to solve the problem for a set of some constant size
  - Know how to merge two solutions of any size together

Merge Sort

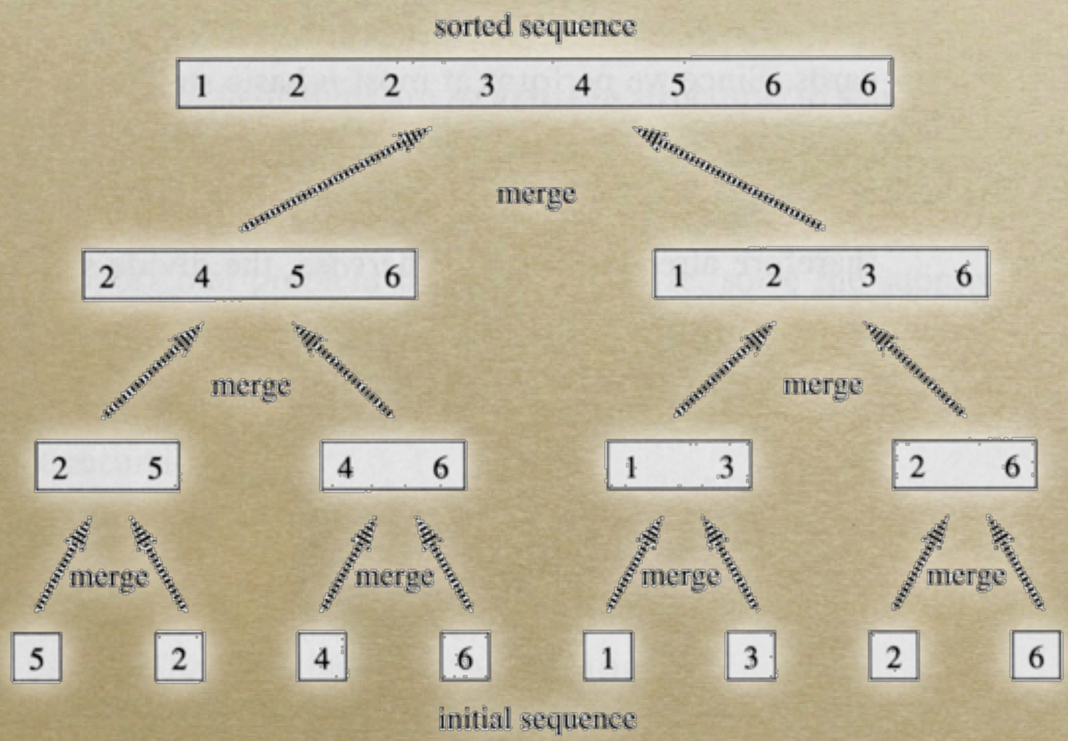
- Sorting: How to sort *n* numbers?
- What do we need to know:
  - How to sort 1 number
  - How to merge two sorted lists of numbers into one

Merge Sort

- Given sorted lists A and B, output a single sorted list O containing all elements of A and B.
- Read the first element of A and B
- Take the smaller of the two
- Remove it from the input list
- Add it to the end of O

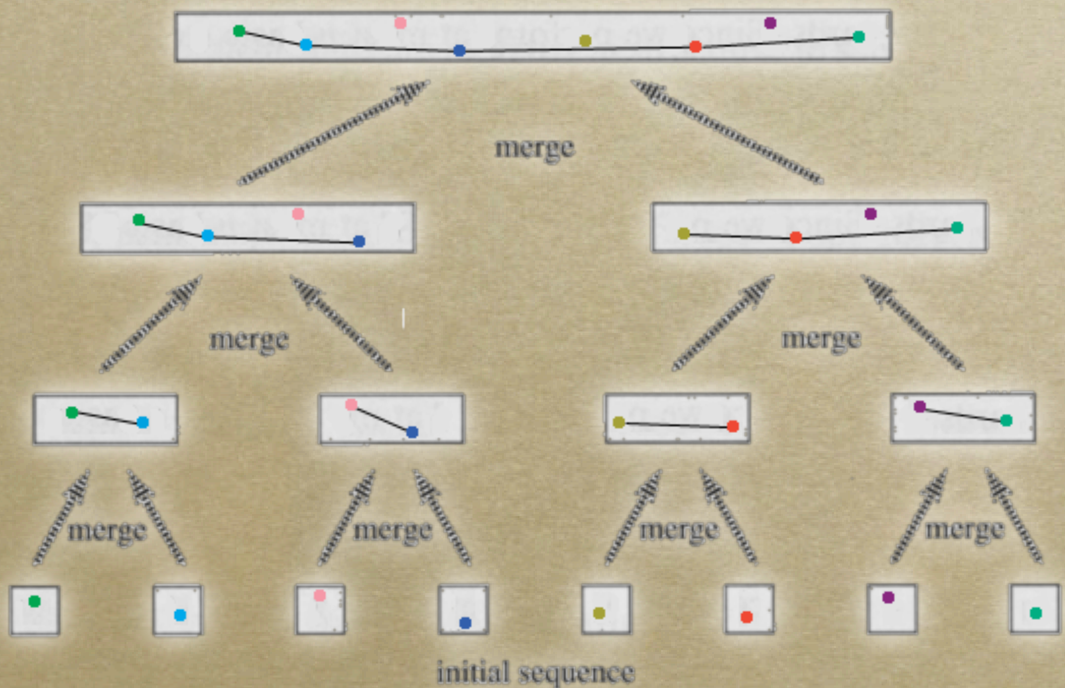
Merge Sort

- Repeat this process until only one sorted sequence is left



Merging Convex Hulls

- We perform similar operations on convex hulls

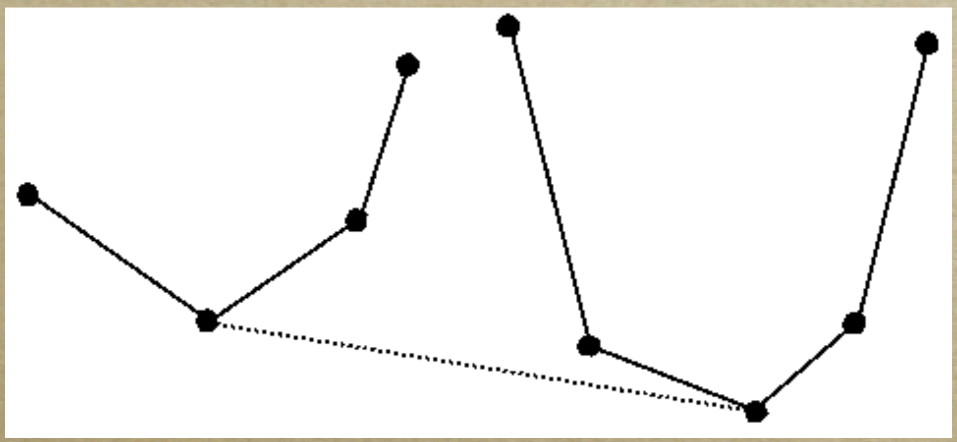


Divide and Conquer 3-d Hull

- We need to know two things:
  - How to construct a 3-d lower hull from a constant sized set of points (e.g. <= 5 points)
  - How to merge two 3-d lower hulls together

Merging 2-d Convex Hulls

- Given two 2-d lower hulls, separated along the *x*-axis
- Find a *bridge* edge between the two hulls



- In 3-d, requires finding many bridge *faces*

Merging 3-d Convex Hulls

- Given two 3-d convex hulls, how do we merge them?
- Project them into 2-d and merge them there
- The third dimension becomes “time”
- 2-d projection point set moves through time

2-d Projection

- For each point p_i, define:
  - $p_i'(t) = (x_i, z_i - ty_i)$
- x-coord stays the same
- y-coord is the 3-d z-coord
- y-coord changes through time, with a constant velocity related to its 3-d y-coord

2-d Projection

- As time t moves from $-\infty$ to ∞, the points will move vertically with constant velocity
- The points start at one extreme and move through to the other
- Over time, the lower hull of the 2-d point set will change

2-d Projection

- By the projection, where $y' = z - ty$
  - A point is a vertex of 3-d lower hull *iff*
  - The 2-d projection of the point:
    - lies on a line $y' = sx + b$ with all other points above it
    - is a vertex of the 2-d lower hull for some time t

Projection Movie

- Think of the 2-d projection as a movie
- By playing the movie of 2-d projection through time, and watching which vertices join the lower hull
- We can construct the 3-d lower hull

Merging

- We want to be able to merge two 3-d hulls
- By merging, we create a 2-d movie
- So, we are given the 3-d hulls, but also their 2-d movies
- Play their movies back together, and merge them together at each time *t*

Demo

Quick Analysis

- Each point's 2-d projection moves with constant velocity, and does not change direction
- Thus during one merge step, it will be added to/removed from the 2-d hull at most once each
- Therefore there are O(n) "events"

Quick Analysis

- The next "event" time for the projection is computed in O(1) time, using the two previous movies
- This gives O(n) time to perform one merge operation
- Thus total run time: $T(n) = 2T(n/2) + O(n) = O(n \log n)$

Remarks

- While we describe the algorithm in terms of time..
- The time is precisely a third coordinate, making the algorithm a space sweep.

Remarks

- We can use this algorithm to answer 3-d extreme point queries
- Given s and t, find a point (x_i, y_i, z_i) minimizing $z_i - sx_i - ty_i$
- Remember the 2-d hull at time t
- Search the lower hull for the point: O(log n) time

Remarks

- Gives a O(log n) time query time for 2-d nearest neighbours problem
- Given a point (a, b) in R^2, find (x_i, y_i):
  - Minimize sqrt($(x_i - a)^2 + (y_i - b)^2$)
  - Minimize $z_i - 2ax_i - 2by_i$ with $z_i = x_i^2 + y_i^2$
- The data structure uses O(n log n) space